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New results of periodic solutions for a kind of Duffing type p-Laplacian equation

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Abstract

By using Mawhin–Manásevich continuation theorem, some new sufficient conditions for the existence and uniqueness of periodic solutions of Duffing type p-Laplacian differential equation are established, which are complement of previously known results.

Keywords: p-Laplacian; Periodic solutions; Duffing equation; Topological degree

1. Introduction

In the last several years, the existence of periodic solutions for the Duffing equation, Rayleigh equation and Liénard type equation has been received a lot of attention. We refer the reader to [1–9] and the references cited therein. However, so far as we know, fewer papers discuss the existence and uniqueness of periodic solutions for Duffing type p-Laplacian differential equation. We only find that Zhang and Li [7] deal with the existence and uniqueness of periodic solutions for Duffing type p-Laplacian differential equation

$$\big(\varphi_p\big(x'(t)\big)\big)' + Cx'(t) + g\big(t, x(t)\big) = e(t), \tag{1}$$

where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, C is a constant, g is a continuous function defined on R^2 and is T-periodic about t, e is a continuous periodic function defined on R with period T, $\int_0^T e(t)\,dt = 0$ and $T > 0$. Zhang and Li [7] provide a sufficient condition for such existence and uniqueness:

Theorem A 1. *Suppose that the following conditions* (A_1), (A_2) *and* (A_3) *hold*:

(A_1) $(g(t,u_1) - g(t,u_2))(u_1 - u_2) < 0$ *for* $t, u_1, u_2 \in R$ *and* $u_1 \neq u_2$.

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(A_2) *$xg(t,x) < 0$ for $|x| > 0$ and $t \in R$.*
(A_3) *There exist positive constants K and M such that*

$$2^{2-p} M T^p < 1, \quad g(t,x) \geqslant -M|x|^{p-1} - K \quad \text{for } x \geqslant 0 \text{ and } t \in R.$$

Then Eq. (1) *has a unique T-periodic solution.*

However, upon examining their proof of Theorem A in [7], we have found that the condition (A_3) can be abandoned. In this paper, we will also discuss the existence and uniqueness of T-periodic solutions of Eq. (1). By using Mawhin–Manásevich continuation theorem, we establish some sufficient conditions for the existence and uniqueness of T-periodic solutions of Eq. (1). These results generalize and improve those in [7].

2. Main result

For convenience, define

$$C_T^1 := \{x \in C^1(R)\colon\ x \text{ is } T\text{-periodic}\},$$

which is a Banach space endowed with the norm $\|.\|$ defined by $\|x\| = |x|_\infty + |x'|_\infty$, for all x, and

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \qquad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|.$$

For the T-periodic boundary value problem

$$\big(\varphi_p\big(x'(t)\big)\big)' = f(t,x,x'), \tag{2}$$

where f is a continuous function and T-periodic in the first variable, we have the following result.

Lemma 1. *(See [9].) Let B be the open in C_T^1 of center 0 and radius r. Assume that the following conditions hold*:

(i) *For each $\lambda \in (0,1)$ the problem $(\phi_p(x'))' = \lambda f(t,x,x')$ has no solution on the board of B.*
(ii) *The continuous function F defined on R by $F(a) = 1/T \int_0^T f(t,a,0)\,dt$ is such that $F(-r)F(r) < 0$.*

Then (2) *has at least one solution in $\bar{B}$.*

By using Lemma 1, we obtain our main results:

Theorem 1. *Let* (A_1) *hold. Suppose that there exists a constant $d \geqslant 0$ such that*

(A_2^*) *$xg(t,x) < 0$ for $|x| > d$ and $t \in R$.*

Then Eq. (1) *has a unique T-periodic solution.*

Proof. Consider the homotopic equation of Eq. (1) as following:

$$\big(\varphi_p\big(x'(t)\big)\big)' + \lambda C x'(t) + \lambda g\big(t, x(t)\big) = \lambda e(t), \quad \lambda \in (0,1). \tag{3}$$

By Lemma 2 in [7], together with (A_1), it is easy to see that Eq. (1) has at most one T-periodic solution. Thus, to prove Theorem 1, it suffices to show that Eq. (1) has at least one T-periodic solution. To do this, we shall apply Lemma 1. Firstly, we will claim that the set of all possible T-periodic solutions of Eq. (3) are bounded.

Let $x(t) \in C_T^1$ be an arbitrary solution of Eq. (3) with period T. By integrating two sides of Eq. (3) over $[0,T]$, and noticing that $x'(0) = x'(T)$, we have

$$\int_0^T g\big(t, x(t)\big)\,dt = 0. \tag{4}$$

As $x(0)=x(T)$, there exists $t_0\in[0,T]$ such that $x'(t_0)=0$, while $\varphi_p(0)=0$ we see

$$\left|\varphi_p\big(x'(t)\big)\right|=\left|\int_{t_0}^{t}\big(\varphi_p\big(x'(s)\big)\big)'\,ds\right|\leqslant\lambda\int_0^T|C|\left|x'(t)\right|dt+\lambda\int_0^T\left|g\big(t,x(t)\big)\right|dt+\lambda\int_0^T\left|e(t)\right|dt,\tag{5}$$

where $t\in[t_0,t_0+T]$.

From (4), there exists $\bar\xi\in[0,T]$ such that

$$g\big(\bar\xi,x(\bar\xi)\big)=0.$$

In view of (A_2^*), we obtain

$$\left|x(\bar\xi)\right|\leqslant d.$$

Then, we have

$$\left|x(t)\right|=\left|x(\bar\xi)+\int_{\bar\xi}^{t}x'(s)\,ds\right|\leqslant d+\int_{\bar\xi}^{t}\left|x'(s)\right|ds,\quad t\in[\bar\xi,\bar\xi+T],$$

and

$$\left|x(t)\right|=\left|x(t-T)\right|=\left|x(\bar\xi)-\int_{t-T}^{\bar\xi}x'(s)\,ds\right|\leqslant d+\int_{t-T}^{\bar\xi}\left|x'(s)\right|ds,\quad t\in[\bar\xi,\bar\xi+T].$$

Combing the above two inequalities, we obtain

$$\begin{aligned}|x|_\infty&=\max_{t\in[0,T]}\left|x(t)\right|=\max_{t\in[\bar\xi,\bar\xi+T]}\left|x(t)\right|\\&\leqslant\max_{t\in[\bar\xi,\bar\xi+T]}\left\{d+\frac12\left(\int_{\bar\xi}^{t}\left|x'(s)\right|ds+\int_{t-T}^{\bar\xi}\left|x'(s)\right|ds\right)\right\}\\&\leqslant d+\frac12\int_0^T\left|x'(s)\right|ds.\end{aligned}\tag{6}$$

Denote

$$E_1=\left\{t\colon\ t\in[0,T],\ \left|x(t)\right|>d\right\},\qquad E_2=\left\{t\colon\ t\in[0,T],\ \left|x(t)\right|\leqslant d\right\}.$$

Since $x(t)$ is T-periodic, multiplying $x(t)$ and (3) and then integrating it from 0 to T, in view of (A_2^*), we get

$$\begin{aligned}\int_0^T\left|x'(t)\right|^p dt&=-\int_0^T\big(\varphi_p\big(x'(t)\big)\big)'x(t)\,dt\\&=\lambda\int_0^T g\big(t,x(t)\big)x(t)\,dt-\lambda\int_0^T e(t)x(t)\,dt\\&=\lambda\int_{E_1}g\big(t,x(t)\big)x(t)\,dt+\lambda\int_{E_2}g\big(t,x(t)\big)x(t)\,dt-\lambda\int_0^T e(t)x(t)\,dt\\&\leqslant\int_0^T\max\left\{\left|g\big(t,x(t)\big)\right|\colon\ t\in R,\ \left|x(t)\right|\leqslant d\right\}\left|x(t)\right|dt+\int_0^T\left|e(t)\right|\left|x(t)\right|dt\\&\leqslant DT|x|_\infty,\end{aligned}\tag{7}$$

where $D=\max\{|g(t,x(t))|\colon\ t\in R,\ |x(t)|\leqslant d\}+|e|_\infty$.

For $x(t) \in C(R,R)$ with $x(t+T) = x(t)$, and $0 < r \leqslant s$, by using Hölder inequality, we obtain

$$\left(\frac{1}{T}\int_0^T |x(t)|^r\,dt\right)^{1/r} \leqslant \left(\frac{1}{T}\left(\int_0^T \left(|x(t)|^r\right)^{\frac{s}{r}}\,dt\right)^{\frac{r}{s}}\left(\int_0^T 1\,dt\right)^{\frac{s-r}{s}}\right)^{1/r} = \left(\frac{1}{T}\int_0^T |x(t)|^s\,dt\right)^{1/s},$$

this implies that

$$|x|_r \leqslant T^{\frac{s-r}{rs}}|x|_s \quad \text{for } 0 < r \leqslant s. \tag{8}$$

Then, in view of (6)–(8), we can get

$$\left(\int_0^T |x'(t)|\,dt\right)^p \leqslant T^{p-1}|x'(t)|_p^p = T^{p-1}\int_0^T |x'(t)|^p\,dt \leqslant T^{p-1}DT|x|_\infty \leqslant T^pD\left(d + \frac{1}{2}\int_0^T |x'(s)|\,ds\right). \tag{9}$$

Since $p > 1$, (9) yields that we can choose some positive constant M_1 such that

$$\int_0^T |x'(s)|\,ds \leqslant M_1, \qquad |x|_\infty \leqslant d + \frac{1}{2}\int_0^T |x'(s)|\,ds \leqslant M_1.$$

In view of (5), we have

$$\begin{aligned}
|x'|_\infty^{p-1} &= \max_{t\in[0,T]}\left\{\left|\varphi_p\big(x'(t)\big)\right|\right\} = \max_{t\in[t_0,t_0+T]}\left\{\left|\int_{t_0}^t \big(\varphi_p\big(x'(s)\big)\big)'\,ds\right|\right\} \\
&\leqslant \int_0^T |C||x'(t)|\,dt + \int_0^T \left|g\big(t,x(t)\big)\right|dt + \int_0^T |e(t)|\,dt \\
&\leqslant |C|M_1 + T\max\left\{\left|g\big(t,x(t)\big)\right|:\ t\in R,\ |x(t)| \leqslant M_1\right\} + T|e|_\infty := M_1^*.
\end{aligned} \tag{10}$$

Thus, we can get some positive constant $M_2 > M_1 + M_1^* + 1$ such that, for all $t \in R$,

$$|x'(t)| \leqslant M_2.$$

Hence taking $r = M_1 + M_2 + d + 1$, we have that the set of all possible T-periodic solutions of Eq. (3) is a subset of B. On the other hand, it is clear that, in our case $F(a) = -1/T\int_0^T g(t,a)\,dt$. From ($A_2^*$) it follows that $F(-r)F(r) < 0$. In consequence we can apply Manásevich–Mawhin continuation theorem to deduce that Eq. (3) has at least one solution in $\bar{B}$. This completes the proof. □

3. Examples and remarks

As an application, let us consider the following equation:

$$\big(\varphi_p x'(t)\big)' + 100x'(t) - \big(67 + \cos^2 t\big)\big(x^9(t) - 12\big) = \cos t, \tag{11}$$

where $p = \sqrt{2}$. We can easily check the conditions (A_1) and (A_2^*) hold. By Theorem 1, Eq. (11) has a unique 2π-periodic solution.

Remark 1. Since $p = \sqrt{2}$, one can easily to see that neither [1–9] nor the references therein can obtain the existence and uniqueness of 2π-periodic solutions for (11). Abandoning the condition (A_3), all the results in Ref. [7] are just the special cases with $d = 0$ of this paper. Moreover, we have corrected Eq. (7) in Ref. [7] as Eq. (5). To sum up, our research results are essentially new and the complement of previous known results.

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